

McGill University  
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE AND APPLIED MATHEMATICS  
Paper ALPHA

16 August, 2016  
1:00 p.m. - 5:00 p.m.

**INSTRUCTIONS:**

- (i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12. Clearly indicate your choice of three questions in each group.
- (ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.
- (iii) Leave enough margins on all sides of the page to permit easy scanning and copying. Each question must be answered on consecutive pages without interruption, and only on the pages on the right side.

This exam comprises this cover and 3 pages of questions.

### Single variable real analysis

Solve any three out of the four questions 1, 2, 3, and 4.

**Problem 1.** Let  $S \subseteq \mathbb{R}$  be such that  $S \neq \emptyset$ ; and  $S$  is bounded below. Let  $s = \inf S$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a non-decreasing continuous function. Prove that

$$\inf f(S) = f(s):$$

**Problem 2.** Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^4 + a_n^2}$$

always converges.

**Problem 3.** Let  $X \subseteq \mathbb{R}$  be compact, and let  $f : X \rightarrow \mathbb{R}$  be continuous. Show that  $f$  is uniformly continuous.

**Problem 4.** Evaluate

(a)

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{e^{x^2} - 1}$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

## Linear Algebra

Solve any three out of the four questions 5, 6, 7, and 8.

### Problem 5.

Let  $L = \{(x; y) \in \mathbb{R}^2 \mid y = x\}$  be a line in  $\mathbb{R}^2$ , and let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the function which maps each point of  $\mathbb{R}^2$  to the point of  $L$  closest to it.

- Show that  $T$  is linear.
- Find the matrix of  $T$  with respect to the standard basis.

**Problem 6.** Let  $\text{Mat}_n$  denote the space of  $n \times n$  matrices with real entries, and let  $\text{Tr} : \text{Mat}_n \rightarrow \mathbb{R}$  denote the trace functional.

- Show that  $\text{Tr}(AB) = \text{Tr}(BA)$  for all  $A, B \in \text{Mat}_n$ .
- Let  $f : \text{Mat}_n \rightarrow \mathbb{R}$  be a linear functional such that  $f(AB) = f(BA)$  for all  $A, B \in \text{Mat}_n$ . Show that  $f$  is a constant multiple of  $\text{Tr}$ .  
(Hint: Consider matrices  $A$  and  $B$  with a single non-zero entry.)

**Problem 7.** Let  $V$  be a complex inner product space, and let  $T : V \rightarrow V$  be linear. Show that  $T$  is unitary if and only if  $T$  maps orthonormal bases to orthonormal bases.

**Problem 8.** For which values  $a; b; c \in \mathbb{R}$  is the matrix

$$\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

diagonalizable over  $\mathbb{R}$ ?

Vector calculus, ODE, and complex analysis

Solve any three out of the four questions 9, 10, 11, and 12.

**Problem 9.** Show that every solution of the constant coefficient equation

$$y'' + py' + qy = 0;$$

tends to 0 as  $x \rightarrow \infty$  if, and only if, the real parts of the roots of the characteristic equation

$$r^2 + pr + q = 0;$$

are *positive*.

**Problem 10.** Find, with full justifications, the maximum and minimum values of

$$f(x; y) = \frac{x}{1 + x^2 + y^2};$$

**Problem 11.** Compute the following integrals.

(a)  $\int_{-1}^1 \int_{-1}^1 x e^{-x^3} dx dy.$

(b)  $\int_D x^2 y^2 dA,$  where  $D = \{(x; y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$  is the unit disk.

**Problem 12.** Show that for any  $\epsilon > 0,$  there is an integer  $n$  such that all the zeros of the function

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots + \frac{1}{n!z^n}; \quad (z \in \mathbb{C});$$

lie in the disk  $D_\epsilon = \{z \in \mathbb{C} : |z| < \epsilon\}.$